General pure multipartite entangled states and the Segre variety

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Abstract

In this paper, we construct a measure of entanglement by generalizing the quadric polynomial of the Segre variety for general multipartite states. We give explicit expressions for general pure three-partite and four-partite states. Moreover, we will discuss and compare this measure of entanglement with the generalized concurrence.

1 Introduction

Quantum entanglement has received a lot of attention during the recent years because of its usefulness in many quantum information and communication tasks such as quantum teleportation and quantum cryptography. However, there is still many open questions concerning the quantification and classification of multipartite states and also its true nature. Thus a deep understanding of this interesting quantum mechanical phenomena could result in construction of new algorithms and protocols for quantum information processing. One widely used measure of entanglement is the so-called concurrence [1]. The connection between concurrence and geometry is found in a map called the Segre embedding [2, 3]. In this paper we will expand our result on the Segre variety [4], which is a quadric space in algebraic geometry, by constructing a measure of entanglement for general pure multipartite states, which also coincides with concurrence of a general pure bipartite and three-partite states [5]. Now, let us start by denoting a general, pure, composite quantum system with m subsystems as $\mathcal{Q} = \mathcal{Q}^p_m(N_1, N_2, \ldots, N_m) = \mathcal{Q}_1\mathcal{Q}_2\cdots\mathcal{Q}_m$, consisting of the pure state $|\Psi\rangle = \sum_{k_1=1}^{N_1}\sum_{k_2=1}^{N_2}\cdots\sum_{k_m=1}^{N_m}\alpha_{k_1,k_2,\ldots,i_m}|k_1, k_2, \ldots, k_m\rangle$ and corresponding to the Hilbert space $\mathcal{H}_{\mathcal{Q}} = \mathcal{H}_{\mathcal{Q}_1}\otimes\mathcal{H}_{\mathcal{Q}_2}\otimes\cdots\otimes\mathcal{H}_{\mathcal{Q}_m}$, where the dimension of the jth Hilbert space is $N_j = \dim(\mathcal{H}_{\mathcal{Q}_j})$. We are going to use this notation throughout this paper. In particular, we denote a pure two-qubit state by $\mathcal{Q}^p_2(2,2)$. Next, let $\rho_{\mathcal{Q}}$ denote a density operator acting on $\mathcal{H}_{\mathcal{Q}}$. The density operator $\rho_{\mathcal{Q}}$ is said to be fully separable, which we will denote by $\rho^{sep}_{\mathcal{Q}}$, with respect to the Hilbert space decomposition, if it can be written as $\rho^{sep}_{\mathcal{Q}} = \sum_{k=1}^{\mathrm{N}} p_k \bigotimes_{j=1}^m \rho^k_{\mathcal{Q}_j}$, $\sum_{k=1}^{\mathrm{N}} p_k = 1$ for some positive integer N, where p_k are positive real numbers and $\rho^k_{\mathcal{Q}_j}$ denotes a density operator on Hilbert space $\mathcal{H}_{\mathcal{Q}_j}$. If $\rho^p_{\mathcal{Q}}$ represents a pure state, then the quantum system is fully separable if $\rho^p_{\mathcal{Q}}$ can be written as $\rho^{sep}_{\mathcal{Q}} = \bigotimes_{j=1}^m \rho_{\mathcal{Q}_j}$, where $\rho_{\mathcal{Q}_j}$ is the density operator on $\mathcal{H}_{\mathcal{Q}_j}$. If a state is not separable, then it is said to be an entangled state. For those readers who are unfamiliar with alge-

braic geometry, we give a short introduction to the basic definition of complex algebraic and projective variety. However, the standard references for the complex projective variety are [6, 7]. Let $\{g_1, g_2, \ldots, g_q\}$ be continuous functions $\mathbf{C}^n \longrightarrow \mathbf{C}$. Then we define a complex space as the set of simultaneous zeroes of the functions

$$\begin{aligned}\mathcal{V}_{\mathbf{C}}(g_1, g_2, \ldots, g_q) = \{(z_1, z_2, \ldots, z_n) \in \mathbf{C}^n : \\ g_i(z_1, z_2, \ldots, z_n) = 0 \ \forall\ 1 \leq i \leq q\}.\end{aligned} \tag{1}$$

The complex space becomes a topological space by giving them the induced topology from $\mathbf{C}^n$. Now, if all g_i are polynomial functions in coordinate functions, then the real (complex) space is called a real (complex) affine variety. A complex projective space $\mathbf{CP}^n$ is defined to be the set of lines through the origin in $\mathbf{C}^{n+1}$, that is,

$$\mathbf{CP}^n = \frac{\mathbf{C}^{n+1} - 0}{(x_1, \ldots, x_{n+1}) \sim (y_1, \ldots, y_{n+1})},\ \lambda \in \mathbf{C} - 0,\ x_i = y_i \forall\ 0 \leq i \leq n. \tag{2}$$

The complex manifold $\mathbf{CP}^1$ of dimension one is a very important one, since as a real manifold it is homeomorphic to the 2-sphere $\mathbf{S}^2$,e.g., the Bloch sphere. Moreover, every complex compact manifold can be embedded in some $\mathbf{CP}^n$. In particular, we can embed a product of two projective spaces into a third one. Let $\{g_1, g_2, \ldots, g_q\}$ be a set of homogeneous polynomials in the coordinates $\{\alpha_1, \alpha_2, \ldots, \alpha_{n+1}\}$ of $\mathbf{C}^{n+1}$. Then the projective variety is defined to be the subset

$$\begin{aligned}\mathcal{V}(g_1, g_2, \ldots, g_q) = \{[\alpha_1, \ldots, \alpha_{n+1}] \in \mathbf{CP}^n : \\ g_i(\alpha_1, \ldots, \alpha_{n+1}) = 0 \ \forall\ 1 \leq i \leq q\}.\end{aligned} \tag{3}$$

We can view the complex affine variety $\mathcal{V}_{\mathbf{C}}(g_1, g_2, \ldots, g_q) \subset \mathbf{C}^{n+1}$ as a complex cone over the projective variety $\mathcal{V}(g_1, g_2, \ldots, g_q)$.

2 Multi-projective variety and a multipartite entanglement measure

In this section, we will review the construction of the Segre variety. Then, we will construct a measure of entanglement for general multipartite states based on a modification of the definition of the Segre variety. This is an extension of our pervious result on construction of a measure of entanglement for general pure multipartite states [4]. We can map the product of space $\mathbf{CP}^{N_1-1} \times \mathbf{CP}^{N_2-1} \times \cdots \times \mathbf{CP}^{N_m-1}$ into a projective variety by its Segre embedding as follows. Let $(\alpha_1, \alpha_2, \ldots, \alpha_{N_i})$ be points defined on the complex projective space $\mathbf{CP}^{N_i-1}$. Then the Segre map

$$\begin{array}{rcl}\mathcal{S}_{N_1,\ldots,N_m} : \mathbf{CP}^{N_1-1} \times \mathbf{CP}^{N_2-1} \times \cdots \times \mathbf{CP}^{N_m-1} & \longrightarrow & \mathbf{CP}^{N_1 N_2 \cdots N_m - 1} \\ ((\alpha_1, \alpha_2, \ldots, \alpha_{N_1}), \ldots, (\alpha_1, \alpha_2, \ldots, \alpha_{N_m})) & \longmapsto & (\ldots, \alpha_{i_1,i_2,\ldots,i_m}, \ldots).\end{array} \tag{4}$$

is well defined for $\alpha_{i_1 i_2 \cdots i_m}$,$1 \leq i_1 \leq N_1, 1 \leq i_2 \leq N_2, \ldots, 1 \leq i_m \leq N_m$ as a homogeneous coordinate-function on $\mathbf{CP}^{N_1 N_2 \cdots N_m - 1}$. Now, let us consider the composite quantum system $\mathcal{Q}^p_m(N_1, N_2, \ldots, N_m)$ and let

$$\mathcal{A} = (\alpha_{i_1,i_2,\ldots,i_m})_{1 \leq i_j \leq N_j}\,, \tag{5}$$

for all $j = 1, 2, \ldots, m$. $\mathcal{A}$ can be realized as the following set $\{(i_1, i_2, \ldots, i_m) : 1 \leq i_j \leq N_j, \forall\ j\}$, in which each point $(i_1, i_2, \ldots, i_m)$ is assigned the value $\alpha_{i_1,i_2,\ldots,i_m}$. This realization of $\mathcal{A}$ is called an m-dimensional box-shape matrix of size $N_1 \times N_2 \times \cdots \times N_m$, where we associate to each such matrix a sub-ring $\mathrm{S}_{\mathcal{A}} = \mathbf{C}[\mathcal{A}] \subset \mathrm{S}$, where S is a commutative ring over the complex number field. For each $j = 1, 2, \ldots, m$, a two-by-two minor about the j-th coordinate of $\mathcal{A}$ is given by

$$\begin{aligned}\mathcal{P}_{k_1,l_1;k_2,l_2;\ldots;k_m,l_m} &= \alpha_{k_1,k_2,\ldots,k_m}\alpha_{l_1,l_2,\ldots,l_m} \\ &\quad -\alpha_{k_1,k_2,\ldots,k_{j-1},l_j,k_{j+1},\ldots,k_m}\alpha_{l_1,l_2,\ldots,l_{j-1},k_j,l_{j+1},\ldots,l_m} \in \mathrm{S}_{\mathcal{A}}.\end{aligned} \tag{6}$$

Then the ideal $\mathcal{I}^m_{\mathcal{A}}$ of $\mathrm{S}_{\mathcal{A}}$ is generated by $\mathcal{P}_{k_1,l_1;k_2,l_2;\ldots;k_m,l_m}$ and describes the separable states in $\mathbf{CP}^{N_1N_2\cdots N_m-1}$ [8]. The image of the Segre embedding $\mathrm{Im}(\mathcal{S}_{N_1,N_2,\ldots,N_m})$, which again is an intersection of families of quadric hypersurfaces in $\mathbf{CP}^{N_1N_2\cdots N_m-1}$, is given by

$$\mathrm{Im}(\mathcal{S}_{N_1,N_2,\ldots,N_m}) = \bigcap_{\forall j} \mathcal{V}\left(\mathcal{C}_{k_1,l_1;k_2,l_2;\ldots;k_m,l_m}\right). \tag{7}$$

Moreover, we can define an entanglement measure for a pure multipartite state as

$$\begin{aligned}\mathcal{E}(\mathcal{Q}^p_m(N_1,\ldots,N_m)) &= \left(\mathcal{N}\sum_{\forall j}|\mathcal{P}_{k_1,l_1;k_2,l_2;\ldots;k_m,l_m}|^2\right)^{\frac{1}{2}} \\ &= (\mathcal{N}\sum_{\forall k_j,l_j,j=1,2,\ldots,m}|\alpha_{k_1,k_2,\ldots,k_m}\alpha_{l_1,l_2,\ldots,l_m} \\ &\quad -\alpha_{k_1,k_2,\ldots,k_{j-1},l_j,k_{j+1},\ldots,k_m}\alpha_{l_1,l_2,\ldots,l_{j-1},k_j,l_{j+1},\ldots,l_m}|^2)^{\frac{1}{2}},\end{aligned} \tag{8}$$

where $\mathcal{N}$ is an arbitrary normalization constant and $j = 1, 2, \ldots, m$. This measure coincides with the generalized concurrence for a general bipartite and three-partite state, but for reasons that we have explained in [4], it fails to quantify the entanglement for $m \geq 4$, whereas it still provides the condition of full separability. However, it is still possible to define an entanglement measure for general multipartite states if we modify the equation (8) in such away that it contains all possible permutations of indices. To do so, we propose a measure of entanglement for general pure multipartite states as

$$\begin{aligned}\mathcal{F}(\mathcal{Q}^p_m(N_1,\ldots,N_m)) &= (\mathcal{N}\sum_{\forall \mathrm{Perm}(\sigma)}\sum_{\forall k_j,l_j,j=1,2,\ldots,m}|\alpha_{k_1,k_2,\ldots,k_m}\alpha_{l_1,l_2,\ldots,l_m} \\ &\quad -\alpha_{\sigma(k_1),\sigma(k_2),\ldots,\sigma(k_m)}\alpha_{\sigma(l_1),\sigma(l_2),\ldots,\sigma(l_m)}|^2)^{\frac{1}{2}},\end{aligned} \tag{9}$$

where $\mathrm{Perm}(\sigma)$ denotes all possible permutations of indices $k_1, k_2, \ldots, k_m$ by $l_1, l_2, \ldots, l_m$, e.g., the first set of permutations is give by equation (6) defining the Segre variety. By construction this measure of entanglement vanishes on product states and it is also invariant under all possible permutations of indices. Note that the first set of permutations defines the Segre variety, but there are also additional products of the complex projective spaces as subspaces of $\mathbf{CP}^{N_1N_2\cdots N_m-1}$ which are defined by other sets of permutations of indices in equation (9).

3 Some examples: three-partite and four-partite states

In this section we will apply this measure of entanglement to three-partite and four-partite states and give explicit expression for the measure of entanglement for these states. We start by the simplest multipartite states, namely three-partite states. Following the recipe in the general expression for multipartite states, we can write the measure of entanglement for such states as

$$\begin{aligned}
\mathcal{F}(\mathcal{Q}_3^p(N_1,N_2,N_3)) &= (\mathcal{N}\sum_{\forall \mathrm{Perm}(\sigma)}\sum_{\forall k_j,l_j,j=1,2,3}|\alpha_{k_1,k_2,k_3}\alpha_{l_1,l_2,l_3} \\
&\quad -\alpha_{\sigma(k_1),\sigma(k_2),\sigma(k_3)}\alpha_{\sigma(l_1),\sigma(l_2),\sigma(l_3)}|^2)^{\frac{1}{2}} \\
&= (\mathcal{N}\sum_{p_1=1}^{3}\sum_{\forall k_j,l_j}|\alpha_{k_1,k_2,k_3}\alpha_{l_1,l_2,l_3}-\alpha_{k_1,l_{p_1},k_3}\alpha_{l_1,k_{p_1},l_3}|^2)^{\frac{1}{2}} \\
&= (\mathcal{N}\sum_{k_1,l_1;k_2,l_2;k_3,l_3}(|\alpha_{k_1,k_2,k_3}\alpha_{l_1,l_2,l_3}-\alpha_{k_1,k_2,l_3}\alpha_{l_1,l_2,k_3}|^2 \\
&\quad +|\alpha_{k_1,k_2,k_3}\alpha_{l_1,l_2,l_3}-\alpha_{k_1,l_2,k_3}\alpha_{l_1,k_2,l_3}|^2) \\
&\quad +|\alpha_{k_1,k_2,k_3}\alpha_{l_1,l_2,l_3}-\alpha_{l_1,k_2,k_3}\alpha_{k_1,l_2,l_3}|^2)^{\frac{1}{2}}.
\end{aligned} \tag{10}$$

This measure of entanglement for three-partite states (10) coincides with generalized concurrence [5]. Moreover, this measure of entanglement measure is equivalent but not equal to our entanglement tensor based on joint POVMs on phase space [9]. Next, we will discuss the measure of entanglement for four-partite states. In this case, we have more then one set of permutations and as we have explained before this is the reason why the measure of entanglement that is directly based on the polynomial that define the Segre variety fails to quantify the entanglement of four-partite states. Now, a measure of entanglement based on the modified Segre variety for four partite states is given by

$$\begin{aligned}
&\mathcal{F}(\mathcal{Q}_4^p(N_1,N_2,N_3,N_4)) = (\mathcal{N}\sum_{\forall \mathrm{Perm}(\sigma)}\sum_{\forall k_j,l_j,j=1,2,3,4}|\alpha_{k_1,k_2,k_3,k_4}\alpha_{l_1,l_2,l_3,l_4} \\
&-\alpha_{\sigma(k_1),\sigma(k_2),\sigma(k_3),\sigma(k_4)}\alpha_{\sigma(l_1),\sigma(l_2),\sigma(l_3),\sigma(l_4)}|^2)^{\frac{1}{2}} \\
&(\mathcal{N}[\sum_{p_1=1}^{4}\sum_{\forall k_j,l_j}|\alpha_{k_1,k_2,k_3,k_4}\alpha_{l_1,l_2,l_3,l_4}-\alpha_{k_1,k_2,l_{p_1},k_4}\alpha_{l_1,l_2,k_{p_1},l_4}|^2 \\
&+\sum_{p_1<p_2}\sum_{\forall k_j,l_j}|\alpha_{k_1,k_2,k_3,k_4}\alpha_{l_1,l_2,l_3,l_4}-\alpha_{k_1,l_{p_1},l_{p_2},k_4}\alpha_{l_1,k_{p_1},k_{p_2},l_4}|^2])^{\frac{1}{2}}.
\end{aligned} \tag{11}$$

The first sum in equation (11) defines the Segre variety, and the second sum gives another product of the complex projective spaces which is still a quadratic subspace of $\mathbf{CP}^{N_1N_2N_3N_4-1}$. For example, for four-qubit states the first set of permutations represented by the first sum gives 112 quadric terms and the second set of permutations, which is represented by the second sum, gives 36 quadric terms. Thus a measure of entanglement for four-qubit states contains 148 terms.

4 Conclusion

In this paper, we have constructed a geometric well-motivated measure of entanglement for general pure multipartite states, based on an extension of the Segre variety. This measure of entanglement works for any pure state and vanishes on multipartite product states. We have also given explicit expressions for our entanglement measure for general pure three-partite and four-partite states. Moreover, we have compared this measure of entanglement with the generalized concurrence.

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